

# Some Open Problems Concerning Orthogonal Polynomials on Fractals and Related Questions

Gökalp Alpan and Alexander Goncharov

## 1 Background and notation

### 1.1 Chebyshev and orthogonal polynomials

Let  $K \subset \mathbb{C}$  be a compact set containing infinitely many points. We use  $\|\cdot\|_{L^\infty(K)}$  to denote the sup-norm on  $K$ ,  $\mathcal{M}_n$  is the set of all monic polynomials of degree  $n$ . The polynomial  $T_{n,K}$  that minimizes  $\|Q_n\|_{L^\infty(K)}$  for  $Q_n \in \mathcal{M}_n$  is called the  $n$ -th *Chebyshev polynomial* on  $K$ .

Let the logarithmic capacity  $\text{Cap}(K)$  be positive. Then we define the  $n$ -th Widom factor for  $K$  by

$$W_n(K) := \|T_{n,K}\|_{L^\infty(K)} / \text{Cap}(K)^n.$$

In what follows we consider unit Borel measures  $\mu$  with non-polar compact support  $\text{supp}(\mu)$  in  $\mathbb{C}$ . The  $n$ -th monic *orthogonal polynomial*  $P_n(z; \mu) = z^n + \dots$  associated with  $\mu$  has the property

$$\|P_n(\cdot; \mu)\|_{L^2(\mu)}^2 = \inf_{Q_n \in \mathcal{M}_n} \int |Q_n(z)|^2 d\mu(z),$$

where  $\|\cdot\|_{L^2(\mu)}$  is the norm in  $L^2(\mu)$ . Then the  $n$ -th *Widom-Hilbert factor* for  $\mu$  is

$$W_n^2(\mu) := \|P_n(\cdot; \mu)\|_{L^2(\mu)} / (\text{Cap}(\text{supp}(\mu)))^n.$$

If  $\text{supp}(\mu) \subset \mathbb{R}$  then a three-term recurrence relation

$$xP_n(x; \mu) = P_{n+1}(x; \mu) + b_{n+1}P_n(x; \mu) + a_n^2P_{n-1}(x; \mu)$$

is valid for  $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . The initial conditions  $P_{-1}(x; \mu) \equiv 0$  and  $P_0(x; \mu) \equiv 1$  generate two bounded sequences  $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$  of *recurrence coefficients* associated with  $\mu$ . Here,  $a_n > 0$ ,  $b_n \in \mathbb{R}$  for  $n \in \mathbb{N}$  and

$$\|P_n(\cdot; \mu)\|_{L^2(\mu)} = a_1 \cdots a_n.$$

A bounded two sided  $\mathbb{C}$ -valued sequence  $(d_n)_{n=-\infty}^\infty$  is called *almost periodic* if the set  $\{(d_{n+k})_{n=-\infty}^\infty : k \in \mathbb{Z}\}$  is precompact in  $l^\infty(\mathbb{Z})$ . A one sided sequence  $(c_n)_{n=1}^\infty$  is called almost periodic if it is the restriction of a two sided almost periodic sequence

to  $\mathbb{N}$ . A sequence  $(e_n)_{n=1}^\infty$  is called *asymptotically almost periodic* if there is an almost periodic sequence  $(e'_n)_{n=1}^\infty$  such that  $|e_n - e'_n| \rightarrow 0$  as  $n \rightarrow \infty$ .

A class of Parreau-Widom sets plays a special role in the recent theory of orthogonal and Chebyshev polynomials. Let  $K$  be a non-polar compact set and  $g_{\mathbb{C} \setminus K}$  denote the Green function for  $\overline{\mathbb{C}} \setminus K$  with a pole at infinity. Suppose  $K$  is regular with respect to the Dirichlet problem, so the set  $\mathcal{C}$  of critical points of  $g_{\mathbb{C} \setminus K}$  is at most countable. Then  $K$  is said to be a *Parreau-Widom* set if  $\sum_{c \in \mathcal{C}} g_{\mathbb{C} \setminus K}(c) < \infty$ . Parreau-Widom sets on  $\mathbb{R}$  have positive Lebesgue measure. For different aspects of such sets, see [9, 16, 24].

A class of regular in the sense of Stahl-Totik measures can be defined by the following condition

$$\lim_{n \rightarrow \infty} W_n(\mu)^{1/n} = 1.$$

For a measure  $\mu$  supported on  $\mathbb{R}$  we use the Lebesgue decomposition of  $\mu$  with respect to the Lebesgue measure:

$$d\mu(x) = f(x)dx + d\mu_s(x).$$

Following [10], let us define the Szegő class  $\text{Sz}(K)$  of measures on a given Parreau-Widom set  $K \subset \mathbb{R}$ . Let  $\mu_K$  be the equilibrium measure on  $K$ . By  $\text{ess supp}(\cdot)$  we denote the essential support of the measure, that is the set of accumulation points of the support. We have  $\text{Cap}(\text{supp}(\mu)) = \text{Cap}(\text{ess supp}(\mu))$ , see Section 1 of [22]. A measure  $\mu$  is in the *Szegő class* of  $K$  if

- (i)  $\text{ess supp}(\mu) = K$ .
- (ii)  $\int_K \log f(x) d\mu_K(x) > -\infty$ . (Szegő condition)
- (iii) the isolated points  $\{x_n\}$  of  $\text{supp}(\mu)$  satisfy  $\sum_n g_{\mathbb{C} \setminus K}(x_n) < \infty$ .

By Theorem 2 in [10] and its proof, (ii) can be replaced by one of the following conditions:

- (ii')  $\limsup_{n \rightarrow \infty} W_n^2(\mu) > 0$ . (Widom condition)
- (ii'')  $\liminf_{n \rightarrow \infty} W_n^2(\mu) > 0$ . (Widom condition 2)

One can show that any  $\mu \in \text{Sz}(K)$  is regular in the sense of Stahl-Totik.

## 1.2 Generalized Julia sets and $K(\gamma)$

Let  $(f_n)_{n=1}^\infty$  be a sequence of rational functions with  $\deg f_n \geq 2$  in  $\overline{\mathbb{C}}$  and  $F_n := f_n \circ f_{n-1} \circ \dots \circ f_1$ . The domain of normality for  $(F_n)_{n=1}^\infty$  in the sense of Montel is called the *Fatou set* for  $(f_n)$ . The complement of the Fatou set in  $\overline{\mathbb{C}}$  is called the *Julia set* for  $(f_n)$ . We denote them by  $F_{(f_n)}$  and  $J_{(f_n)}$  respectively. These sets were considered first

in [12]. In particular, if  $f_n = f$  for some fixed rational function  $f$  for all  $n$  then  $F_{(f)}$  and  $J_{(f)}$  are used instead. To distinguish this last case, the word *autonomous* is used in the literature.

Suppose  $f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j$  where  $d_n \geq 2$  and  $a_{n,d_n} \neq 0$  for all  $n \in \mathbb{N}$ . Following [8], we say that  $(f_n)$  is a *regular polynomial sequence* (write  $(f_n) \in \mathcal{R}$ ) if positive constants  $A_1, A_2, A_3$  exist such that for all  $n \in \mathbb{N}$  we have the following three conditions:

$$\begin{aligned} |a_{n,d_n}| &\geq A_1 \\ |a_{n,j}| &\leq A_2 |a_{n,d_n}| \text{ for } j = 0, 1, \dots, d_n - 1 \\ \log |a_{n,d_n}| &\leq A_3 \cdot d_n \end{aligned}$$

For such polynomial sequences, by [8],  $J_{(f_n)}$  is a regular compact set in  $\mathbb{C}$ . In addition,  $\text{Cap}(J_{(f_n)}) > 0$  and  $J_{(f_n)}$  is the boundary of

$$\mathcal{A}_{(f_n)}(\infty) := \{z \in \overline{\mathbb{C}} : F_n(z) \text{ goes locally uniformly to } \infty\}.$$

The following construction is from [13]. Let  $\gamma := (\gamma_k)_{k=1}^\infty$  be a sequence provided that  $0 < \gamma_k < 1/4$  holds for all  $k \in \mathbb{N}$  and  $\gamma_0 := 1$ . Let  $f_1(z) = 2z(z-1)/\gamma_1 + 1$  and  $f_n(z) = \frac{1}{2\gamma_n}(z^2 - 1) + 1$  for  $n > 1$ . Then  $K(\gamma) := \cap_{s=1}^\infty F_s^{-1}([-1, 1])$  is a Cantor set on  $\mathbb{R}$ . Furthermore,  $F_s^{-1}([-1, 1]) \subset F_t^{-1}([-1, 1]) \subset [0, 1]$  whenever  $s > t$ .

Also we use an expanded version of this set. For a sequence  $\gamma$  as above, let  $f_n(z) = \frac{1}{2\gamma_n}(z^2 - 1) + 1$  for  $n \in \mathbb{N}$ . Then  $K_1(\gamma) := \cap_{s=1}^\infty F_s^{-1}([-1, 1]) \subset [-1, 1]$  and  $F_s^{-1}([-1, 1]) \subset F_t^{-1}([-1, 1]) \subset [-1, 1]$  provided that  $s > t$ . It is a Cantor set. If there is a  $c$  with  $0 < c < \gamma_k$  for all  $k$  then  $(f_n) \in \mathcal{R}$  and  $J_{(f_n)} = K_1(\gamma)$ , see [5]. If  $\gamma_1 = \dots = \gamma_k$  for all  $k \in \mathbb{N}$  then  $K_1(\gamma)$  is an autonomous polynomial Julia set.

### 1.3 Hausdorff measure

A function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is called a *dimension function* if it is increasing, continuous and  $h(0) = 0$ . Given a set  $E \subset \mathbb{C}$ , its  *$h$ -Hausdorff measure* is defined as

$$\Lambda_h(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum h(r_j) : E \subset \bigcup B(z_j, r_j) \text{ with } r_j \leq \delta \right\},$$

where  $B(z, r)$  is the open ball of radius  $r$  centered at  $z$ . For a dimension function  $h$ , a set  $K \subset \mathbb{C}$  is an  *$h$ -set* if  $0 < \Lambda_h(K) < \infty$ . To denote the Hausdorff measure for  $h(t) = t^\alpha$ ,  $\Lambda_\alpha$  is used. *Hausdorff dimension* of  $K$  is defined as  $\inf\{\alpha \geq 0 : \Lambda_\alpha(K) = 0\}$ .

## 2 Smoothness of Green functions and Markov Factors

The next set of problems is concerned with the smoothness properties of the Green function  $g_{\mathbb{C} \setminus K}$  near compact set  $K$  and related questions. We suppose that  $K$  is regular with respect to the Dirichlet problem, so the function  $g_{\mathbb{C} \setminus K}$  is continuous throughout  $\mathbb{C}$ . The next problem was posed in [13].

**Problem 1.** Given modulus of continuity  $\omega$ , find a compact set  $K$  such that the modulus of continuity  $\omega(g_{\mathbb{C} \setminus K}, \cdot)$  is similar to  $\omega$ .

Here, one can consider similarity as coincidence of moduli on some null sequence or in the sense of weak equivalence:  $\exists C_1, C_2$  such that

$$C_1 \omega(\delta) \leq \omega(g_{\mathbb{C} \setminus K}, \delta) \leq C_2 \omega(\delta)$$

for sufficiently small positive  $\delta$ .

We guess that a set  $K(\gamma)$  from [13] is a candidate for the desired  $K$  provided a suitable choice of the parameters. We recall that, for many moduli of continuity, the corresponding Green's functions were given in [13], whereas the characterization of optimal smoothness for  $g_{\mathbb{C} \setminus K(\gamma)}$  is presented in [[5], Th.6.3].

A stronger version of the problem is about pointwise estimation of the Green function:

**Problem 2.** Given modulus of continuity  $\omega$ , find a compact set  $K$  such that

$$C_1 \omega(\delta) \leq g_{\mathbb{C} \setminus K}(z) \leq C_2 \omega(\delta)$$

for  $\delta = \text{dist}(z, K) \leq \delta_0$ , where  $C_1, C_2$  and  $\delta_0$  do not depend on  $z$ .

In the most important case we get a problem about “two-sided Hölder” Green function, which was posed by A. Volberg on his seminar (quoted with permission):

**Problem 3.** Find a compact set  $K$  on the line such that for some  $\alpha > 0$  and constants  $C_1, C_2$ , if  $\delta = \text{dist}(z, K)$  is small enough then

$$C_1 \delta^\alpha \leq g_{\mathbb{C} \setminus K}(z) \leq C_2 \delta^\alpha. \quad (1)$$

Clearly, a closed analytic curve gives a solution for sets on the plane.

If  $K \subset \mathbb{R}$  satisfies (1), then  $K$  is of Cantor-type. Indeed, if interior of  $K$  (with respect to  $\mathbb{R}$ ) is not empty, let  $(a, b) \subset K$ , then  $g_{\mathbb{C} \setminus K}$  has *Lip* 1 behavior near the point  $(a+b)/2$ . On the other hand, near endpoints of  $K$  the function  $g_{\mathbb{C} \setminus K}$  cannot be better than *Lip*  $1/2$ .

By the Bernstein-Walsh inequality, smoothness properties of Green functions are closely related with a character of maximal growth of polynomials outside the corresponding compact sets, which, in turn, allows to evaluate Markov's factors for the sets. Recall that, for a fixed  $n \in \mathbb{N}$  and (infinite) compact set  $K$ , the  $n$ -th Markov factor  $M_n(K)$  is the norm of operator of differentiation in the space of holomorphic polynomials  $\mathcal{P}_n$  with the uniform norm on  $K$ . In particular, the Hölder smoothness (the right inequality in (1)) implies Markov's property of the set  $K$  (a polynomial growth rate of  $M_n(K)$ ). The problem about inverse implication (see e.g [20]) has attracted attention of many researches.

By W. Pleśniak [20], any Markov set  $K \subset \mathbb{R}^d$  has the extension property (*EP*), which means that there exists a continuous linear extension operator from the space of Whitney functions  $\mathcal{E}(K)$  to the space of infinitely differentiable functions on  $\mathbb{R}^d$ . We guess that there is some extremal growth rate of  $M_n$  which implies the lack of *EP*. Recently it was shown in [15] that there is no complete characterization of *EP* in terms of growth rate of Markov's factors. Namely, two sets were presented,  $K_1$  with *EP* and  $K_2$  without it, such that  $M_n(K_1)$  grows essentially faster than  $M_n(K_2)$  as  $n \rightarrow \infty$ . Thus there exists non-empty zone of uncertainty where the growth rate of  $M_n(K)$  is not related with *EP* of the set  $K$ .

**Problem 4.** Characterize the growth rates of Markov's factors that define the boundaries of the zone of uncertainty for the extension property.

### 3 Orthogonal polynomials

One of the most interesting problems concerning orthogonal polynomials on Cantor sets on  $\mathbb{R}$  is the character of periodicity of recurrence coefficients. It was conjectured in p. 123 of [7] that if  $f$  is a non-linear polynomial such that  $J(f)$  is a totally disconnected subset of  $\mathbb{R}$  then the recurrence coefficients for  $\mu_{J(f)}$  are almost periodic. This is still an open problem. In [6], the authors conjectured that the recurrence coefficients for  $\mu_{K(\gamma)}$  are asymptotically almost periodic for any  $\gamma$ . It may be hoped that a more general and slightly weaker version of Bellissard's conjecture can be valid.

**Problem 5.** Let  $(f_n)$  be a regular polynomial sequence such that  $J(f_n)$  is a Cantor-type subset of the real line. Prove that the recurrence coefficients for  $\mu_{J(f_n)}$  are asymptotically almost periodic.

For a measure  $\mu$  which is supported on  $\mathbb{R}$ , let  $Z_n(\mu) := \{x : P_n(x; \mu) = 0\}$ . We define  $U_n(\mu)$  by

$$U_n(\mu) := \inf_{\substack{x, x' \in Z_n(\mu) \\ x \neq x'}} |x - x'|.$$

In [18] Krüger and Simon gave a lower bound for  $U_n(\mu)$  where  $\mu$  is the Cantor-Lebesgue measure of the (translated and scaled) Cantor ternary set. In [17], it was shown that Markov's inequality and spacing of the zeros of orthogonal polynomials are somewhat related.

Let  $\gamma = (\gamma_k)_{k=1}^\infty$  and  $n \in \mathbb{N}$  with  $n > 1$  be given and define  $\delta_k = \gamma_0 \cdots \gamma_k$  for all  $k \in \mathbb{N}_0$ . Let  $s$  be the integer satisfying  $2^{s-1} \leq n < 2^s$ . By [2],

$$\delta_{s+2} \leq U_n(\mu_{K(\gamma)}) \leq \frac{\pi^2}{4} \cdot \delta_{s-2}$$

holds. In particular, if there is a number  $c$  such that  $0 < c < \gamma_k < 1/4$  holds for all  $k \in \mathbb{N}$  then, by [2], we have

$$c^2 \cdot \delta_s \leq U_n(\mu_{K(\gamma)}) \leq \frac{\pi^2}{4c^2} \cdot \delta_s. \quad (2)$$

By [13], at least for small sets  $K(\gamma)$ , we have  $M_{2^s}(K(\gamma)) \sim 2/\delta_s$ , where the symbol  $\sim$  means the strong equivalence.

**Problem 6.** Let  $K$  be a non-polar compact subset of  $\mathbb{R}$ . Is there a general relation between the zero spacing of orthogonal polynomials for  $\mu_K$  and smoothness of  $g_{\mathbb{C} \setminus K}$ ? Is there a relation between the zero spacing of  $\mu_K$  and the Markov factors?

As mentioned in section 1, the Szegő condition and the Widom condition are equivalent for Parreau-Widom sets. Let  $K$  be a Parreau-Widom set. Let  $\mu$  be a measure such that  $\text{ess supp}(\mu) = K$  and the isolated points  $\{x_n\}$  of  $\text{supp}(\mu)$  satisfy  $\sum_n g_{\mathbb{C} \setminus K}(x_n) < \infty$ . Then, as it is discussed in Section 6 of [4], the Szegő condition is equivalent to the condition

$$\int_K \log(d\mu/d\mu_K) d\mu_K(x) > -\infty. \quad (3)$$

This condition is also equivalent to the Widom condition under these assumptions.

It was shown in [1] that  $\inf_{n \in \mathbb{N}} W_n(\mu_K) \geq 1$  for non-polar compact  $K \subset \mathbb{R}$ . Thus the Szegő condition in the form (3) and the Widom condition are related on arbitrary non-polar sets.

**Problem 7.** Let  $K$  be a non-polar compact subset of  $\mathbb{R}$  which is regular with respect to the Dirichlet problem. Let  $\mu$  be a measure such that  $\text{ess supp}(\mu) = K$ . Assume that the isolated points  $\{x_n\}$  of  $\text{supp}(\mu)$  satisfy  $\sum_n g_{\mathbb{C} \setminus K}(x_n) < \infty$ . If the condition (3) is valid for  $\mu$  is it necessarily true that the Widom condition or the Widom condition 2 holds? Conversely, does the Widom condition imply (3)?

It was proved in [11] that if  $K$  is a Parreau-Widom set which is a subset of  $\mathbb{R}$  then  $(W_n(K))_{n=1}^\infty$  is bounded above. On the other hand,  $(W_n(K))_{n=1}^\infty$  is unbounded for some Cantor-type sets, see e.g. [14].

**Problem 8.** Is it possible to find a regular non-polar compact subset  $K$  of  $\mathbb{R}$  which is not Parreau-Widom but  $(W_n(K))_{n=1}^\infty$  is bounded? If  $K$  has zero Lebesgue measure then is it true that  $(W_n(K))_{n=1}^\infty$  is unbounded? We can ask the same problems if we replace  $(W_n(K))_{n=1}^\infty$  by  $(W_n^2(\mu_K))_{n=1}^\infty$  above.

Let  $T_N$  be a real polynomial of degree  $N$  with  $N \geq 2$  such that it has  $N$  real and simple zeros  $x_1 < \dots < x_n$  and  $N - 1$  critical points  $y_1 < \dots < y_{n-1}$  with  $|T_N(y_i)| \geq 1$  for each  $i \in \{1, \dots, N - 1\}$ . We call such a polynomial admissible. If  $K = T_N^{-1}([-1, 1])$  for an admissible polynomial  $T_N$  then  $K$  is called a  $T$ -set. The following result is well known, see e.g. [23].

**Theorem 1.** Let  $K = \cup_{j=1}^n [\alpha_j, \beta_j]$  be a disjoint union of  $n$  intervals such that  $\alpha_1$  is the leftmost end point. Then  $K$  is a  $T$ -set if and only if  $\mu_K([\alpha_1, c])$  is in  $\mathbb{Q}$  for all  $c \in \mathbb{R} \setminus K$ .

For  $K(\gamma)$ , it is known that  $\mu_{K(\gamma)}([0, c]) \in \mathbb{Q}$  if  $c \in \mathbb{R} \setminus K(\gamma)$ , see Section 4 in [2].

**Problem 9.** Let  $K$  be a regular non-polar compact subset of  $\mathbb{R}$  and  $\alpha$  be the leftmost end point of  $K$ . Let  $\mu_K([\alpha, c]) \in \mathbb{Q}$  for all  $c \in \mathbb{R} \setminus K$ . What can we say about  $K$ ? Is it necessarily a polynomial generalized Julia set? Does this imply that there is a sequence of admissible polynomials  $(f_n)_{n=1}^\infty$  such that  $(F_n^{-1}[-1, 1])_{n=1}^\infty$  is a decreasing sequence of sets such that  $K = \cap_{n=1}^\infty F_n^{-1}[-1, 1]$ ?

## 4 Hausdorff measures

It is valid for a wide class of Cantor sets that the equilibrium measure and the corresponding Hausdorff measure on this set are mutually singular, see e.g. [19].

Let  $\gamma = (\gamma_k)_{k=1}^\infty$  with  $0 < \gamma_k < 1/32$  satisfy  $\sum_{k=1}^\infty \gamma_k < \infty$ . This implies that  $K(\gamma)$  has Hausdorff dimension 0. In [3], the authors constructed a dimension function  $h_\gamma$  that makes  $K(\gamma)$  an  $h$ -set. Provided also that  $K(\gamma)$  is not polar it was shown that there is a  $C > 0$  such that for any Borel set  $B$ ,  $C^{-1} \cdot \mu_{K(\gamma)}(B) < \Lambda_{h_\gamma}(B) < C \cdot \mu_{K(\gamma)}(B)$  and in particular the equilibrium measure and  $\Lambda_{h_\gamma}$  restricted to  $K(\gamma)$  are mutually absolutely continuous. In [15], it was shown by the authors that indeed these two measures coincide. To the best of our knowledge, this is the first example of a subset of  $\mathbb{R}$  such that the equilibrium measure is a Hausdorff measure restricted to the set.

**Problem 10.** Let  $K$  be a non-polar compact subset of  $\mathbb{R}$  such that  $\mu_K$  is equal to a Hausdorff measure restricted to  $K$ . Is it necessarily true that the Hausdorff dimension of  $K$  is 0?

Hausdorff dimension of a unit Borel measure  $\mu$  supported on  $\mathbb{C}$  is defined by  $\dim(\mu) := \inf\{HD(K) : \mu(K) = 1\}$  where  $HD(\cdot)$  denotes Hausdorff dimension of the given set. For polynomial Julia sets which are totally disconnected there is a formula for  $\dim(\mu_{J(f)})$ , see e.g. p. 23 in [19] and p.176-177 in [21].

**Problem 11.** Is it possible to find simple formulas for  $\dim(\mu_{J(f_n)})$  where  $(f_n)$  is a regular polynomial sequence?

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The authors are partially supported by a grant from Tübitak: 115F199